

The theory of matrices and determinants

Why it is important to understand: The theory of matrices and determinants

Matrices are used to solve problems in electronics, optics, quantum mechanics, statics, robotics, linear programming, optimisation, genetics, and much more. Matrix calculus is a mathematical tool used in connection with linear equations, linear transformations, systems of differential equations, and so on, and is vital for calculating forces, vectors, tensions, masses, loads and a lot of other factors that must be accounted for in engineering to ensure safe and resource-efficient structure. Electrical and mechanical engineers, chemists, biologists and scientists all need knowledge of matrices to solve problems. In computer graphics, matrices are used to project a three-dimensional image on to a two-dimensional screen, and to create realistic motion. Matrices are therefore very important in solving engineering problems.

At the end of this chapter, you should be able to:

- understand matrix notation
- add, subtract and multiply 2 by 2 and 3 by 3 matrices
- recognise the unit matrix
- calculate the determinant of a 2 by 2 matrix
- determine the inverse (or reciprocal) of a 2 by 2 matrix
- calculate the determinant of a 3 by 3 matrix
- determine the inverse (or reciprocal) of a 3 by 3 matrix

24.1 Matrix notation

Matrices and determinants are mainly used for the solution of linear simultaneous equations. The theory of matrices and determinants is dealt with in this chapter and this theory is then used in Chapter 25 to solve simultaneous equations.

The coefficients of the variables for linear simultaneous equations may be shown in matrix form.

The coefficients of x and y in the simultaneous equations

$$x + 2y = 3$$

$$4x - 5y = 6$$

become $\begin{pmatrix} 1 & 2 \\ 4 & -5 \end{pmatrix}$ in matrix notation.

Similarly, the coefficients of p , q and r in the equations

$$1.3p - 2.0q + r = 7$$

$$3.7p + 4.8q - 7r = 3$$

$$4.1p + 3.8q + 12r = -6$$

become $\begin{pmatrix} 1.3 & -2.0 & 1 \\ 3.7 & 4.8 & -7 \\ 4.1 & 3.8 & 12 \end{pmatrix}$ in matrix form.

The numbers within a matrix are called an **array** and the coefficients forming the array are called the **elements** of the matrix. The number of rows in a matrix is usually specified by m and the number of columns by n and a matrix referred to as an ' m by n ' matrix.

Thus, $\begin{pmatrix} 2 & 3 & 6 \\ 4 & 5 & 7 \end{pmatrix}$ is a '2 by 3' matrix. Matrices cannot be expressed as a single numerical value, but they can often be simplified or combined, and unknown element values can be determined by comparison methods. Just as there are rules for addition, subtraction, multiplication and division of numbers in arithmetic, rules for these operations can be applied to matrices and the rules of matrices are such that they obey most of those governing the algebra of numbers.

24.2 Addition, subtraction and multiplication of matrices

(i) Addition of matrices

Corresponding elements in two matrices may be added to form a single matrix.

Problem 1. Add the matrices

(a) $\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix}$ and $\begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix}$

(a) Adding the corresponding elements gives:

$$\begin{aligned} & \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 2 + (-3) & -1 + 0 \\ -7 + 7 & 4 + (-4) \end{pmatrix} \\ &= \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

(b) Adding the corresponding elements gives:

$$\begin{aligned} & \begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix} + \begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 3+2 & 1+7 & -4+(-5) \\ 4+(-2) & 3+1 & 1+0 \\ 1+6 & 4+3 & -3+4 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 8 & -9 \\ 2 & 4 & 1 \\ 7 & 7 & 1 \end{pmatrix} \end{aligned}$$

(ii) Subtraction of matrices

If A is a matrix and B is another matrix, then $(A - B)$ is a single matrix formed by subtracting the elements of B from the corresponding elements of A .

Problem 2. Subtract

(a) $\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$ from $\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix}$ from $\begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix}$

To find matrix A minus matrix B , the elements of B are taken from the corresponding elements of A . Thus:

$$\begin{aligned} \text{(a)} \quad & \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} - \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 2 - (-3) & -1 - 0 \\ -7 - 7 & 4 - (-4) \end{pmatrix} \\ &= \begin{pmatrix} 5 & -1 \\ -14 & 8 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & \begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix} - \begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 3-2 & 1-7 & -4-(-5) \\ 4-(-2) & 3-1 & 1-0 \\ 1-6 & 4-3 & -3-4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -6 & 1 \\ 6 & 2 & 1 \\ -5 & 1 & -7 \end{pmatrix} \end{aligned}$$

Problem 3. If

$$A = \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}, B = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} \text{ and}$$

$$C = \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix} \text{ find } A + B - C$$

$$A + B = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \quad (\text{from Problem 1})$$

$$\begin{aligned} \text{Hence, } A + B - C &= \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix} \\ &= \begin{pmatrix} -1-1 & -1-0 \\ 0-(-2) & 0-(-4) \end{pmatrix} \\ &= \begin{pmatrix} -2 & -1 \\ 2 & 4 \end{pmatrix} \end{aligned}$$

Alternatively $A + B - C$

$$\begin{aligned} &= \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix} \\ &= \begin{pmatrix} -3+2-1 & 0+(-1)-0 \\ 7+(-7)-(-2) & -4+4-(-4) \end{pmatrix} \\ &= \begin{pmatrix} -2 & -1 \\ 2 & 4 \end{pmatrix} \text{ as obtained previously} \end{aligned}$$

(iii) Multiplication

When a matrix is multiplied by a number, called **scalar multiplication**, a single matrix results in which each element of the original matrix has been multiplied by the number.

$$\begin{aligned} \text{Problem 4. If } A &= \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}, \\ B &= \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix} \text{ find} \\ 2A - 3B + 4C \end{aligned}$$

For scalar multiplication, each element is multiplied by the scalar quantity, hence

$$\begin{aligned} 2A &= 2 \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ 14 & -8 \end{pmatrix} \\ 3B &= 3 \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -21 & 12 \end{pmatrix} \\ \text{and } 4C &= 4 \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -8 & -16 \end{pmatrix} \end{aligned}$$

Hence $2A - 3B + 4C$

$$\begin{aligned} &= \begin{pmatrix} -6 & 0 \\ 14 & -8 \end{pmatrix} - \begin{pmatrix} 6 & -3 \\ -21 & 12 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ -8 & -16 \end{pmatrix} \\ &= \begin{pmatrix} -6-6+4 & 0-(-3)+0 \\ 14-(-21)+(-8) & -8-12+(-16) \end{pmatrix} \\ &= \begin{pmatrix} -8 & 3 \\ 27 & -36 \end{pmatrix} \end{aligned}$$

When a matrix A is multiplied by another matrix B , a single matrix results in which elements are obtained from the sum of the products of the corresponding rows of A and the corresponding columns of B .

Two matrices A and B may be multiplied together, provided the number of elements in the rows of matrix A are equal to the number of elements in the columns of matrix B . In general terms, when multiplying a matrix of dimensions (m by n) by a matrix of dimensions (n by r), the resulting matrix has dimensions (m by r). Thus a 2 by 3 matrix multiplied by a 3 by 1 matrix gives a matrix of dimensions 2 by 1.

$$\begin{aligned} \text{Problem 5. If } A &= \begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix} \text{ and } B = \begin{pmatrix} -5 & 7 \\ -3 & 4 \end{pmatrix} \\ \text{find } A \times B \end{aligned}$$

$$\text{Let } A \times B = C \text{ where } C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

C_{11} is the sum of the products of the first row elements of A and the first column elements of B taken one at a time,

$$\text{i.e. } C_{11} = (2 \times (-5)) + (3 \times (-3)) = -19$$

C_{12} is the sum of the products of the first row elements of A and the second column elements of B , taken one at a time,

$$\text{i.e. } C_{12} = (2 \times 7) + (3 \times 4) = 26$$

C_{21} is the sum of the products of the second row elements of A and the first column elements of B , taken one at a time,

$$\text{i.e. } C_{21} = (1 \times (-5)) + (-4 \times (-3)) = 7$$

Finally, C_{22} is the sum of the products of the second row elements of A and the second column elements of B , taken one at a time,

$$\text{i.e. } C_{22} = (1 \times 7) + ((-4) \times 4) = -9$$

$$\text{Thus, } A \times B = \begin{pmatrix} -19 & 26 \\ 7 & -9 \end{pmatrix}$$

Problem 6. Simplify

$$\begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

The sum of the products of the elements of each row of the first matrix and the elements of the second matrix, (called a **column matrix**), are taken one at a time. Thus:

$$\begin{aligned} & \begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} (3 \times 2) + (4 \times 5) + (0 \times (-1)) \\ (-2 \times 2) + (6 \times 5) + (-3 \times (-1)) \\ (7 \times 2) + (-4 \times 5) + (1 \times (-1)) \end{pmatrix} \\ &= \begin{pmatrix} 26 \\ 29 \\ -7 \end{pmatrix} \end{aligned}$$

Problem 7. If $A = \begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix}$ and

$$B = \begin{pmatrix} 2 & -5 \\ 5 & -6 \\ -1 & -7 \end{pmatrix}, \text{ find } A \times B$$

The sum of the products of the elements of each row of the first matrix and the elements of each column of the second matrix are taken one at a time. Thus:

$$\begin{aligned} & \begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & -5 \\ 5 & -6 \\ -1 & -7 \end{pmatrix} \\ &= \begin{pmatrix} [(3 \times 2) + (4 \times 5) + (0 \times (-1))] & [(3 \times (-5)) + (4 \times (-6)) + (0 \times (-7))] \\ [(-2 \times 2) + (6 \times 5) + (-3 \times (-1))] & [(-2 \times (-5)) + (6 \times (-6)) + (-3 \times (-7))] \\ [(7 \times 2) + (-4 \times 5) + (1 \times (-1))] & [(7 \times (-5)) + (-4 \times (-6)) + (1 \times (-7))] \end{pmatrix} \\ &= \begin{pmatrix} 26 & -39 \\ 29 & -5 \\ -7 & -18 \end{pmatrix} \end{aligned}$$

Problem 8. Determine

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 2 & 0 \\ 1 & 3 & 2 \\ 3 & 2 & 0 \end{pmatrix}$$

The sum of the products of the elements of each row of the first matrix and the elements of each column of the second matrix are taken one at a time. Thus:

$$\begin{aligned} & \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 2 & 0 \\ 1 & 3 & 2 \\ 3 & 2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} [(1 \times 2) + (0 \times 1) + (3 \times 3)] & [(1 \times 2) + (0 \times 3) + (3 \times 2)] & [(1 \times 0) + (0 \times 2) + (3 \times 0)] \\ [(2 \times 2) + (1 \times 1) + (2 \times 3)] & [(2 \times 2) + (1 \times 3) + (2 \times 2)] & [(2 \times 0) + (1 \times 2) + (2 \times 0)] \\ [(1 \times 2) + (3 \times 1) + (1 \times 3)] & [(1 \times 2) + (3 \times 3) + (1 \times 2)] & [(1 \times 0) + (3 \times 2) + (1 \times 0)] \end{pmatrix} \\ &= \begin{pmatrix} 11 & 8 & 0 \\ 11 & 11 & 2 \\ 8 & 13 & 6 \end{pmatrix} \end{aligned}$$

In algebra, the commutative law of multiplication states that $a \times b = b \times a$. For matrices, this law is only true in a few special cases, and in general $A \times B$ is **not** equal to $B \times A$

Problem 9. If $A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$ and

$$B = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \text{ show that } A \times B \neq B \times A$$

$$\begin{aligned} A \times B &= \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} [(2 \times 2) + (3 \times 0)] & [(2 \times 3) + (3 \times 1)] \\ [(1 \times 2) + (0 \times 0)] & [(1 \times 3) + (0 \times 1)] \end{pmatrix} \\ &= \begin{pmatrix} 4 & 9 \\ 2 & 3 \end{pmatrix} \\ B \times A &= \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} [(2 \times 2) + (3 \times 1)] & [(2 \times 3) + (3 \times 0)] \\ [(0 \times 2) + (1 \times 1)] & [(0 \times 3) + (1 \times 0)] \end{pmatrix} \\ &= \begin{pmatrix} 7 & 6 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Since $\begin{pmatrix} 4 & 9 \\ 2 & 3 \end{pmatrix} \neq \begin{pmatrix} 7 & 6 \\ 1 & 0 \end{pmatrix}$, then $A \times B \neq B \times A$

Now try the following Practice Exercise

Practice Exercise 105 Addition, subtraction and multiplication of matrices (Answers on page 846)

In Problems 1 to 13, the matrices A to K are:

$$A = \begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} \quad B = \begin{pmatrix} 5 & 2 \\ -1 & 6 \end{pmatrix}$$

$$C = \begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix}$$

$$D = \begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

$$E = \begin{pmatrix} 3 & 6 & 2 \\ 5 & -3 & 7 \\ -1 & 0 & 2 \end{pmatrix}$$

$$F = \begin{pmatrix} 3.1 & 2.4 & 6.4 \\ -1.6 & 3.8 & -1.9 \\ 5.3 & 3.4 & -4.8 \end{pmatrix} \quad G = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$$

$$H = \begin{pmatrix} -2 \\ 5 \end{pmatrix} \quad J = \begin{pmatrix} 4 \\ -11 \\ 7 \end{pmatrix} \quad K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In Problems 1 to 12, perform the matrix operation stated.

1. $A + B$
2. $D + E$
3. $A - B$
4. $A + B - C$
5. $5A + 6B$
6. $2D + 3E - 4F$
7. $A \times H$
8. $A \times B$
9. $A \times C$
10. $D \times J$
11. $E \times K$

12. $D \times F$

13. Show that $A \times C \neq C \times A$

24.3 The unit matrix

A **unit matrix**, I , is one in which all elements of the leading diagonal (\) have a value of 1 and all other elements have a value of 0. Multiplication of a matrix by I is the equivalent of multiplying by 1 in arithmetic.

24.4 The determinant of a 2 by 2 matrix

The **determinant** of a 2 by 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as $(ad - bc)$

The elements of the determinant of a matrix are written between vertical lines. Thus, the determinant of $\begin{pmatrix} 3 & -4 \\ 1 & 6 \end{pmatrix}$ is written as $\begin{vmatrix} 3 & -4 \\ 1 & 6 \end{vmatrix}$ and is equal to $(3 \times 6) - (-4 \times 1)$, i.e. $18 - (-4)$ or 22. Hence the determinant of a matrix can be expressed as a single numerical value, i.e. $\begin{vmatrix} 3 & -4 \\ 1 & 6 \end{vmatrix} = 22$

Problem 10. Determine the value of

$$\begin{vmatrix} 3 & -2 \\ 7 & 4 \end{vmatrix}$$

$$\begin{vmatrix} 3 & -2 \\ 7 & 4 \end{vmatrix} = (3 \times 4) - (-2 \times 7) \\ = 12 - (-14) = 26$$

Problem 11. Evaluate $\begin{vmatrix} (1+j) & j2 \\ -j3 & (1-j4) \end{vmatrix}$

$$\begin{vmatrix} (1+j) & j2 \\ -j3 & (1-j4) \end{vmatrix} = (1+j)(1-j4) - (j2)(-j3) \\ = 1 - j4 + j - j^2 4 + j^2 6 \\ = 1 - j4 + j - (-4) + (-6) \\ \text{since from Chapter 22, } j^2 = -1 \\ = 1 - j4 + j + 4 - 6 \\ = -1 - j3$$

Problem 12. Evaluate $\begin{vmatrix} 5\angle 30^\circ & 2\angle -60^\circ \\ 3\angle 60^\circ & 4\angle -90^\circ \end{vmatrix}$

$$\begin{aligned} \begin{vmatrix} 5\angle 30^\circ & 2\angle -60^\circ \\ 3\angle 60^\circ & 4\angle -90^\circ \end{vmatrix} &= (5\angle 30^\circ)(4\angle -90^\circ) \\ &\quad - (2\angle -60^\circ)(3\angle 60^\circ) \\ &= (20\angle -60^\circ) - (6\angle 0^\circ) \\ &= (10 - j17.32) - (6 + j0) \\ &= \mathbf{(4 - j17.32) \text{ or } 17.78\angle -77^\circ} \end{aligned}$$

Now try the following Practice Exercise

Practice Exercise 106 2 by 2 determinants
(Answers on page 846)

1. Calculate the determinant of $\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix}$
2. Calculate the determinant of $\begin{pmatrix} -2 & 5 \\ 3 & -6 \end{pmatrix}$
3. Calculate the determinant of $\begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix}$
4. Evaluate $\begin{vmatrix} j2 & -j3 \\ (1+j) & j \end{vmatrix}$
5. Evaluate $\begin{vmatrix} 2\angle 40^\circ & 5\angle -20^\circ \\ 7\angle -32^\circ & 4\angle -117^\circ \end{vmatrix}$
6. Given matrix $A = \begin{pmatrix} (x-2) & 6 \\ 2 & (x-3) \end{pmatrix}$, determine values of x for which $|A| = 0$

24.5 The inverse or reciprocal of a 2 by 2 matrix

The inverse of matrix A is A^{-1} such that $A \times A^{-1} = I$, the unit matrix.

Let matrix A be $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and let the inverse matrix, A^{-1}

be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then, since $A \times A^{-1} = I$,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiplying the matrices on the left-hand side, gives

$$\begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Equating corresponding elements gives:

$$b+2d=0, \text{ i.e. } b=-2d$$

$$\text{and } 3a+4c=0, \text{ i.e. } a=-\frac{4}{3}c$$

Substituting for a and b gives:

$$\begin{pmatrix} -\frac{4}{3}c+2c & -2d+2d \\ 3\left(-\frac{4}{3}c\right)+4c & 3(-2d)+4d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{i.e. } \begin{pmatrix} \frac{2}{3}c & 0 \\ 0 & -2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

showing that $\frac{2}{3}c=1$, i.e. $c=\frac{3}{2}$ and $-2d=1$, i.e. $d=-\frac{1}{2}$

Since $b=-2d$, $b=1$ and since $a=-\frac{4}{3}c$, $a=-2$

Thus the inverse of matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that is,

$$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

There is, however, a **quicker method of obtaining the inverse** of a 2 by 2 matrix.

For any matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ the inverse may be obtained by:

- (i) interchanging the positions of p and s ,
- (ii) changing the signs of q and r , and
- (iii) multiplying this new matrix by the reciprocal of the determinant of $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$

Thus the inverse of matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is

$$\frac{1}{4-6} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

as obtained previously.

Problem 13. Determine the inverse of

$$\begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix}$$

The inverse of matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is obtained by interchanging the positions of p and s , changing the signs of q and r and multiplying by the reciprocal of the determinant $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$. Thus, the inverse of

$$\begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix} = \frac{1}{(3 \times 4) - (-2 \times 7)} \begin{pmatrix} 4 & 2 \\ -7 & 3 \end{pmatrix} \\ = \frac{1}{26} \begin{pmatrix} 4 & 2 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ -\frac{7}{26} & \frac{3}{26} \end{pmatrix}$$

Now try the following Practice Exercise

Practice Exercise 107 The inverse of 2 by 2 matrices (Answers on page 846)

- Determine the inverse of $\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix}$
- Determine the inverse of $\begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{3}{5} \end{pmatrix}$
- Determine the inverse of $\begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix}$

24.6 The determinant of a 3 by 3 matrix

- (i) The **minor** of an element of a 3 by 3 matrix is the value of the 2 by 2 determinant obtained by covering up the row and column containing that element.

Thus for the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ the minor of element 4 is obtained by covering the row (4 5 6) and the column $\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$, leaving the 2 by

2 determinant $\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}$, i.e. the minor of element 4 is $(2 \times 9) - (3 \times 8) = -6$

- (ii) The sign of a minor depends on its position within the matrix, the sign pattern being $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.

Thus the signed-minor of element 4 in the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ is $-\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = -(-6) = 6$

The signed-minor of an element is called the **cofactor** of the element.

- (iii) **The value of a 3 by 3 determinant is the sum of the products of the elements and their cofactors of any row or any column of the corresponding 3 by 3 matrix.**

There are thus six different ways of evaluating a 3×3 determinant – and all should give the same value.

Problem 14. Find the value of

$$\begin{vmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{vmatrix}$$

The value of this determinant is the sum of the products of the elements and their cofactors, of any row or of any column. If the second row or second column is selected, the element 0 will make the product of the element and its cofactor zero and reduce the amount of arithmetic to be done to a minimum.

Supposing a second row expansion is selected. The minor of 2 is the value of the determinant remaining when the row and column containing the 2 (i.e. the second row and the first column), is covered up.

Thus the cofactor of element 2 is $\begin{vmatrix} 4 & -1 \\ -3 & -2 \end{vmatrix}$ i.e. -11

The sign of element 2 is minus (see (ii) above), hence the cofactor of element 2, (the signed-minor) is $+11$

Similarly the minor of element 7 is $\begin{vmatrix} 3 & 4 \\ 1 & -3 \end{vmatrix}$ i.e. -13 ,

and its cofactor is $+13$. Hence the value of the sum of the products of the elements and their cofactors is $2 \times 11 + 0 + 7 \times 13$, i.e.,

$$\begin{vmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{vmatrix} = 2(11) + 0 + 7(13) = \mathbf{113}$$

The same result will be obtained whichever row or column is selected. For example, the third column expansion is

$$(-1) \begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} - 7 \begin{vmatrix} 3 & 4 \\ 1 & -3 \end{vmatrix} + (-2) \begin{vmatrix} 3 & 4 \\ 2 & 0 \end{vmatrix}$$

$$= 6 + 91 + 16 = \mathbf{113}, \text{ as obtained previously.}$$

Problem 15. Evaluate $\begin{vmatrix} 1 & 4 & -3 \\ -5 & 2 & 6 \\ -1 & -4 & 2 \end{vmatrix}$

Using the first row: $\begin{vmatrix} 1 & 4 & -3 \\ -5 & 2 & 6 \\ -1 & -4 & 2 \end{vmatrix}$

$$= 1 \begin{vmatrix} 2 & 6 \\ -4 & 2 \end{vmatrix} - 4 \begin{vmatrix} -5 & 6 \\ -1 & 2 \end{vmatrix} + (-3) \begin{vmatrix} -5 & 2 \\ -1 & -4 \end{vmatrix}$$

$$= (4 + 24) - 4(-10 + 6) - 3(20 + 2)$$

$$= 28 + 16 - 66 = \mathbf{-22}$$

Using the second column: $\begin{vmatrix} 1 & 4 & -3 \\ -5 & 2 & 6 \\ -1 & -4 & 2 \end{vmatrix}$

$$= -4 \begin{vmatrix} -5 & 6 \\ -1 & 2 \end{vmatrix} + 2 \begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix} - (-4) \begin{vmatrix} 1 & -3 \\ -5 & 6 \end{vmatrix}$$

$$= -4(-10 + 6) + 2(2 - 3) + 4(6 - 15)$$

$$= 16 - 2 - 36 = \mathbf{-22}$$

Problem 16. Determine the value of

$$\begin{vmatrix} j2 & (1+j) & 3 \\ (1-j) & 1 & j \\ 0 & j4 & 5 \end{vmatrix}$$

Using the first column, the value of the determinant is:

$$(j2) \begin{vmatrix} 1 & j \\ j4 & 5 \end{vmatrix} - (1-j) \begin{vmatrix} (1+j) & 3 \\ j4 & 5 \end{vmatrix} + (0) \begin{vmatrix} (1+j) & 3 \\ 1 & j \end{vmatrix}$$

$$= j2(5 - j^2 4) - (1-j)(5 + j5 - j12) + 0$$

$$= j2(9) - (1-j)(5 - j7)$$

$$= j18 - [5 - j7 - j5 + j^2 7]$$

$$= j18 - [-2 - j12]$$

$$= j18 + 2 + j12 = \mathbf{2 + j30 \text{ or } 30.07 \angle 86.19^\circ}$$

Now try the following Practice Exercise

Practice Exercise 108 3 by 3 determinants
(Answers on page 847)

1. Find the matrix of minors of

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

2. Find the matrix of cofactors of

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

3. Calculate the determinant of

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

4. Evaluate $\begin{vmatrix} 8 & -2 & -10 \\ 2 & -3 & -2 \\ 6 & 3 & 8 \end{vmatrix}$

5. Calculate the determinant of

$$\begin{pmatrix} 3.1 & 2.4 & 6.4 \\ -1.6 & 3.8 & -1.9 \\ 5.3 & 3.4 & -4.8 \end{pmatrix}$$

6. Evaluate $\begin{vmatrix} j2 & 2 & j \\ (1+j) & 1 & -3 \\ 5 & -j4 & 0 \end{vmatrix}$

7. Evaluate $\begin{vmatrix} 3 \angle 60^\circ & j2 & 1 \\ 0 & (1+j) & 2 \angle 30^\circ \\ 0 & 2 & j5 \end{vmatrix}$

8. Find the eigenvalues λ that satisfy the following equations:

(a) $\begin{vmatrix} (2-\lambda) & 2 \\ -1 & (5-\lambda) \end{vmatrix} = 0$

(b) $\begin{vmatrix} (5-\lambda) & 7 & -5 \\ 0 & (4-\lambda) & -1 \\ 2 & 8 & (-3-\lambda) \end{vmatrix} = 0$

(You may need to refer to Chapter 1, pages 10–14, for the solution of cubic equations).

24.7 The inverse or reciprocal of a 3 by 3 matrix

The **adjoint** of a matrix A is obtained by:

- forming a matrix B of the cofactors of A , and
- transposing** matrix B to give B^T , where B^T is the matrix obtained by writing the rows of B as the columns of B^T . Then **adj** $A = B^T$

The **inverse of matrix** A , A^{-1} is given by

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

where $\text{adj } A$ is the adjoint of matrix A and $|A|$ is the determinant of matrix A .

Problem 17. Determine the inverse of the matrix

$$\begin{pmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{pmatrix}$$

The inverse of matrix A , $A^{-1} = \frac{\text{adj } A}{|A|}$

The adjoint of A is found by:

- obtaining the matrix of the cofactors of the elements, and
- transposing this matrix.

The cofactor of element 3 is $+\begin{vmatrix} 0 & 7 \\ -3 & -2 \end{vmatrix} = 21$

The cofactor of element 4 is $-\begin{vmatrix} 2 & 7 \\ 1 & -2 \end{vmatrix} = 11$, and so on.

The matrix of cofactors is $\begin{pmatrix} 21 & 11 & -6 \\ 11 & -5 & 13 \\ 28 & -23 & -8 \end{pmatrix}$

The transpose of the matrix of cofactors, i.e. the adjoint of the matrix, is obtained by writing the rows as columns,

and is $\begin{pmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{pmatrix}$

From Problem 14, the determinant of $\begin{vmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{vmatrix}$ is 113

Hence the inverse of $\begin{pmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{pmatrix}$ is

$$\frac{\begin{pmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{pmatrix}}{113} \text{ or } \frac{1}{113} \begin{pmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{pmatrix}$$

Problem 18. Find the inverse of

$$\begin{pmatrix} 1 & 5 & -2 \\ 3 & -1 & 4 \\ -3 & 6 & -7 \end{pmatrix}$$

Inverse = $\frac{\text{adjoint}}{\text{determinant}}$

The matrix of cofactors is $\begin{pmatrix} -17 & 9 & 15 \\ 23 & -13 & -21 \\ 18 & -10 & -16 \end{pmatrix}$

The transpose of the matrix of cofactors (i.e. the adjoint) is $\begin{pmatrix} -17 & 23 & 18 \\ 9 & -13 & -10 \\ 15 & -21 & -16 \end{pmatrix}$

The determinant of $\begin{pmatrix} 1 & 5 & -2 \\ 3 & -1 & 4 \\ -3 & 6 & -7 \end{pmatrix}$

$$= 1(7 - 24) - 5(-21 + 12) - 2(18 - 3) \\ = -17 + 45 - 30 = -2$$

Hence the inverse of $\begin{pmatrix} 1 & 5 & -2 \\ 3 & -1 & 4 \\ -3 & 6 & -7 \end{pmatrix}$

$$= \frac{\begin{pmatrix} -17 & 23 & 18 \\ 9 & -13 & -10 \\ 15 & -21 & -16 \end{pmatrix}}{-2} \\ = \begin{pmatrix} 8.5 & -11.5 & -9 \\ -4.5 & 6.5 & 5 \\ -7.5 & 10.5 & 8 \end{pmatrix}$$

Now try the following Practice Exercise

Practice Exercise 109 The inverse of a 3 by 3 matrix (Answers on page 847)

- Write down the transpose of

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

2. Write down the transpose of

$$\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$$

3. Determine the adjoint of

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

4. Determine the adjoint of

$$\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$$

5. Find the inverse of

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

6. Find the inverse of
- $\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$

Applications of matrices and determinants

Why it is important to understand: Applications of matrices and determinants

As mentioned previously, matrices are used to solve problems, for example, in electrical circuits, optics, quantum mechanics, statics, robotics, genetics, and much more, and for calculating forces, vectors, tensions, masses, loads and a lot of other factors that must be accounted for in engineering. In the main, matrices and determinants are used to solve a system of simultaneous linear equations. The simultaneous solution of multiple equations finds its way into many common engineering problems. In fact, modern structural engineering analysis techniques are all about solving systems of equations simultaneously. Eigenvalues and eigenvectors, which are based on matrix theory, are very important in engineering and science. For example, car designers analyse eigenvalues in order to damp out the noise in a car, eigenvalue analysis is used in the design of car stereo systems, eigenvalues can be used to test for cracks and deformities in a solid, and oil companies use eigenvalues analysis to explore land for oil.

At the end of this chapter, you should be able to:

- solve simultaneous equations in two and three unknowns using matrices
- solve simultaneous equations in two and three unknowns using determinants
- solve simultaneous equations using Cramer's rule
- solve simultaneous equations using Gaussian elimination
- determine the eigenvalues of a 2 by 2 and 3 by 3 matrix
- determine the eigenvectors of a 2 by 2 and 3 by 3 matrix

25.1 Solution of simultaneous equations by matrices

- (a) The procedure for solving linear simultaneous equations in **two unknowns using matrices** is:

- (i) write the equations in the form

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

- (ii) write the matrix equation corresponding to these equations,

$$\text{i.e. } \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

- (iii) determine the inverse matrix of $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$

$$\text{i.e. } \frac{1}{a_1b_2 - b_1a_2} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}$$

(from Chapter 24)

- (iv) multiply each side of (ii) by the inverse matrix, and

- (v) solve for x and y by equating corresponding elements.

Problem 1. Use matrices to solve the simultaneous equations:

$$3x + 5y - 7 = 0 \quad (1)$$

$$4x - 3y - 19 = 0 \quad (2)$$

- (i) Writing the equations in the $a_1x + b_1y = c$ form gives:

$$3x + 5y = 7$$

$$4x - 3y = 19$$

- (ii) The matrix equation is

$$\begin{pmatrix} 3 & 5 \\ 4 & -3 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 19 \end{pmatrix}$$

- (iii) The inverse of matrix $\begin{pmatrix} 3 & 5 \\ 4 & -3 \end{pmatrix}$ is

$$\frac{1}{3 \times (-3) - 5 \times 4} \begin{pmatrix} -3 & -5 \\ -4 & 3 \end{pmatrix}$$

$$\text{i.e. } \begin{pmatrix} \frac{3}{29} & \frac{5}{29} \\ \frac{4}{29} & \frac{-3}{29} \end{pmatrix}$$

- (iv) Multiplying each side of (ii) by (iii) and remembering that $A \times A^{-1} = I$, the unit matrix, gives:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{29} & \frac{5}{29} \\ \frac{4}{29} & \frac{-3}{29} \end{pmatrix} \times \begin{pmatrix} 7 \\ 19 \end{pmatrix}$$

$$\text{Thus } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{21}{29} + \frac{95}{29} \\ \frac{28}{29} - \frac{57}{29} \end{pmatrix}$$

$$\text{i.e. } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

- (v) By comparing corresponding elements:

$$x = 4 \quad \text{and} \quad y = -1$$

Checking:

equation (1),

$$3 \times 4 + 5 \times (-1) - 7 = 0 = \text{RHS}$$

equation (2),

$$4 \times 4 - 3 \times (-1) - 19 = 0 = \text{RHS}$$

- (b) The procedure for solving linear simultaneous equations in **three unknowns using matrices** is:

- (i) write the equations in the form

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

- (ii) write the matrix equation corresponding to these equations, i.e.

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

- (iii) determine the inverse matrix of

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \quad (\text{see Chapter 24})$$

- (iv) multiply each side of (ii) by the inverse matrix, and

- (v) solve for x , y and z by equating the corresponding elements.

Problem 2. Use matrices to solve the simultaneous equations:

$$x + y + z - 4 = 0 \quad (1)$$

$$2x - 3y + 4z - 33 = 0 \quad (2)$$

$$3x - 2y - 2z - 2 = 0 \quad (3)$$

- (i) Writing the equations in the $a_1x + b_1y + c_1z = d_1$ form gives:

$$x + y + z = 4$$

$$2x - 3y + 4z = 33$$

$$3x - 2y - 2z = 2$$

- (ii) The matrix equation is

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & -2 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 33 \\ 2 \end{pmatrix}$$

- (iii) The inverse matrix of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & -2 \end{pmatrix}$$

is given by

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

The adjoint of A is the transpose of the matrix of the cofactors of the elements (see Chapter 24). The matrix of cofactors is

$$\begin{pmatrix} 14 & 16 & 5 \\ 0 & -5 & 5 \\ 7 & -2 & -5 \end{pmatrix}$$

and the transpose of this matrix gives

$$\text{adj } A = \begin{pmatrix} 14 & 0 & 7 \\ 16 & -5 & -2 \\ 5 & 5 & -5 \end{pmatrix}$$

The determinant of A , i.e. the sum of the products of elements and their cofactors, using a first row expansion is

$$\begin{aligned} & 1 \begin{vmatrix} -3 & 4 \\ -2 & -2 \end{vmatrix} - 1 \begin{vmatrix} 2 & 4 \\ 3 & -2 \end{vmatrix} + 1 \begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} \\ &= (1 \times 14) - (1 \times (-16)) + (1 \times 5) = 35 \end{aligned}$$

Hence the inverse of A ,

$$A^{-1} = \frac{1}{35} \begin{pmatrix} 14 & 0 & 7 \\ 16 & -5 & -2 \\ 5 & 5 & -5 \end{pmatrix}$$

- (iv) Multiplying each side of (ii) by (iii), and remembering that $A \times A^{-1} = I$, the unit matrix, gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{35} \begin{pmatrix} 14 & 0 & 7 \\ 16 & -5 & -2 \\ 5 & 5 & -5 \end{pmatrix} \times \begin{pmatrix} 4 \\ 33 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{35} \begin{pmatrix} (14 \times 4) + (0 \times 33) + (7 \times 2) \\ (16 \times 4) + ((-5) \times 33) + ((-2) \times 2) \\ (5 \times 4) + (5 \times 33) + ((-5) \times 2) \end{pmatrix}$$

$$= \frac{1}{35} \begin{pmatrix} 70 \\ -105 \\ 175 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

- (v) By comparing corresponding elements, $x=2$, $y=-3$, $z=5$, which can be checked in the original equations.

Now try the following Practice Exercise

Practice Exercise 110 Solving simultaneous equations using matrices (Answers on page 847)

In Problems 1 to 5 use **matrices** to solve the simultaneous equations given.

- $3x + 4y = 0$
 $2x + 5y + 7 = 0$
- $2p + 5q + 14.6 = 0$
 $3.1p + 1.7q + 2.06 = 0$
- $x + 2y + 3z = 5$
 $2x - 3y - z = 3$
 $-3x + 4y + 5z = 3$
- $3a + 4b - 3c = 2$
 $-2a + 2b + 2c = 15$
 $7a - 5b + 4c = 26$
- $p + 2q + 3r + 7.8 = 0$
 $2p + 5q - r - 1.4 = 0$
 $5p - q + 7r - 3.5 = 0$

6. In two closed loops of an electrical circuit, the currents flowing are given by the simultaneous equations:

$$I_1 + 2I_2 + 4 = 0$$

$$5I_1 + 3I_2 - 1 = 0$$

Use matrices to solve for I_1 and I_2

7. The relationship between the displacement s , velocity v , and acceleration a , of a piston is given by the equations:

$$s + 2v + 2a = 4$$

$$3s - v + 4a = 25$$

$$3s + 2v - a = -4$$

Use matrices to determine the values of s , v and a

8. In a mechanical system, acceleration $\ddot{x}$, velocity $\dot{x}$ and distance x are related by the simultaneous equations:

$$3.4\ddot{x} + 7.0\dot{x} - 13.2x = -11.39$$

$$-6.0\ddot{x} + 4.0\dot{x} + 3.5x = 4.98$$

$$2.7\ddot{x} + 6.0\dot{x} + 7.1x = 15.91$$

Use matrices to find the values of $\ddot{x}$, $\dot{x}$ and x

25.2 Solution of simultaneous equations by determinants

- (a) When solving linear simultaneous equations in **two unknowns using determinants:**

- (i) write the equations in the form

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

and then

- (ii) the solution is given by

$$\frac{x}{D_x} = \frac{-y}{D_y} = \frac{1}{D}$$

$$\text{where } D_x = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

i.e. the determinant of the coefficients left when the x -column is covered up,

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

i.e. the determinant of the coefficients left when the y -column is covered up,

$$\text{and } D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

i.e. the determinant of the coefficients left when the constants-column is covered up.

Problem 3. Solve the following simultaneous equations using determinants:

$$3x - 4y = 12$$

$$7x + 5y = 6.5$$

Following the above procedure:

$$(i) \quad 3x - 4y - 12 = 0$$

$$7x + 5y - 6.5 = 0$$

$$(ii) \quad \frac{x}{\begin{vmatrix} -4 & -12 \\ 5 & -6.5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & -12 \\ 7 & -6.5 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & -4 \\ 7 & 5 \end{vmatrix}}$$

$$\begin{aligned} \text{i.e. } \frac{x}{(-4)(-6.5) - (-12)(5)} &= \frac{-y}{(3)(-6.5) - (-12)(7)} = \frac{1}{(3)(5) - (-4)(7)} \\ &= \frac{-y}{(3)(-6.5) - (-12)(7)} \\ &= \frac{1}{(3)(5) - (-4)(7)} \end{aligned}$$

$$\text{i.e. } \frac{x}{26 + 60} = \frac{-y}{-19.5 + 84} = \frac{1}{15 + 28}$$

$$\text{i.e. } \frac{x}{86} = \frac{-y}{64.5} = \frac{1}{43}$$

$$\text{Since } \frac{x}{86} = \frac{1}{43} \text{ then } x = \frac{86}{43} = 2$$

and since

$$\frac{-y}{64.5} = \frac{1}{43} \text{ then } y = -\frac{64.5}{43} = -1.5$$

Problem 4. The velocity of a car, accelerating at uniform acceleration a between two points, is given by $v = u + at$, where u is its velocity when passing the first point and t is the time taken to pass between the two points. If $v = 21$ m/s when $t = 3.5$ s and $v = 33$ m/s when $t = 6.1$ s, use determinants to find the values of u and a , each correct to 4 significant figures.

Substituting the given values in $v = u + at$ gives:

$$21 = u + 3.5a \quad (1)$$

$$33 = u + 6.1a \quad (2)$$

(i) The equations are written in the form

$$a_1x + b_1y + c_1 = 0$$

i.e. $u + 3.5a - 21 = 0$

and $u + 6.1a - 33 = 0$

(ii) The solution is given by

$$\frac{u}{D_u} = \frac{-a}{D_a} = \frac{1}{D}$$

where D_u is the determinant of coefficients left when the u column is covered up,

$$\begin{aligned} \text{i.e. } D_u &= \begin{vmatrix} 3.5 & -21 \\ 6.1 & -33 \end{vmatrix} \\ &= (3.5)(-33) - (-21)(6.1) \\ &= 12.6 \end{aligned}$$

$$\begin{aligned} \text{Similarly, } D_a &= \begin{vmatrix} 1 & -21 \\ 1 & -33 \end{vmatrix} \\ &= (1)(-33) - (-21)(1) \\ &= -12 \end{aligned}$$

$$\begin{aligned} \text{and } D &= \begin{vmatrix} 1 & 3.5 \\ 1 & 6.1 \end{vmatrix} \\ &= (1)(6.1) - (3.5)(1) = 2.6 \end{aligned}$$

$$\text{Thus } \frac{u}{12.6} = \frac{-a}{-12} = \frac{1}{26}$$

$$\text{i.e. } u = \frac{12.6}{2.6} = 4.846 \text{ m/s}$$

$$\begin{aligned} \text{and } a &= \frac{12}{2.6} = 4.615 \text{ m/s}^2, \\ &\text{each correct to 4 significant figures.} \end{aligned}$$

Problem 5. Applying Kirchhoff's laws to an electric circuit results in the following equations:

$$\begin{aligned} (9 + j12)I_1 - (6 + j8)I_2 &= 5 \\ -(6 + j8)I_1 + (8 + j3)I_2 &= (2 + j4) \end{aligned}$$

Solve the equations for I_1 and I_2

Following the procedure:

$$\begin{aligned} \text{(i)} \quad (9 + j12)I_1 - (6 + j8)I_2 &= 5 \\ -(6 + j8)I_1 + (8 + j3)I_2 &= (2 + j4) \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \frac{I_1}{\begin{vmatrix} -(6 + j8) & -5 \\ (8 + j3) & -(2 + j4) \end{vmatrix}} &= \frac{-I_2}{\begin{vmatrix} (9 + j12) & -5 \\ -(6 + j8) & -(2 + j4) \end{vmatrix}} \\ &= \frac{1}{\begin{vmatrix} (9 + j12) & -(6 + j8) \\ -(6 + j8) & (8 + j3) \end{vmatrix}} \\ &= \frac{1}{(-20 + j40) + (40 + j15)} \\ &= \frac{-I_2}{(30 - j60) - (30 + j40)} \\ &= \frac{1}{(36 + j123) - (-28 + j96)} \\ \frac{I_1}{20 + j55} &= \frac{-I_2}{-j100} = \frac{1}{64 + j27} \\ \text{Hence } I_1 &= \frac{20 + j55}{64 + j27} \\ &= \frac{58.52 \angle 70.02^\circ}{69.46 \angle 22.87^\circ} = 0.84 \angle 47.15^\circ \text{ A} \\ \text{and } I_2 &= \frac{100 \angle 90^\circ}{69.46 \angle 22.87^\circ} \\ &= 1.44 \angle 67.13^\circ \text{ A} \end{aligned}$$

(b) When solving simultaneous equations in **three unknowns using determinants:**

(i) Write the equations in the form

$$a_1x + b_1y + c_1z + d_1 = 0$$

$$a_2x + b_2y + c_2z + d_2 = 0$$

$$a_3x + b_3y + c_3z + d_3 = 0$$

and then

(ii) the solution is given by

$$\frac{x}{D_x} = \frac{-y}{D_y} = \frac{z}{D_z} = \frac{-1}{D}$$

where D_x is $\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$

i.e. the determinant of the coefficients obtained by covering up the x column.

D_y is $\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}$

i.e., the determinant of the coefficients obtained by covering up the y column.

D_z is $\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

i.e. the determinant of the coefficients obtained by covering up the z column.

and D is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

i.e. the determinant of the coefficients obtained by covering up the constants column.

Problem 6. A d.c. circuit comprises three closed loops. Applying Kirchhoff's laws to the closed loops gives the following equations for current flow in milliamperes:

$$2I_1 + 3I_2 - 4I_3 = 26$$

$$I_1 - 5I_2 - 3I_3 = -87$$

$$-7I_1 + 2I_2 + 6I_3 = 12$$

Use determinants to solve for I_1 , I_2 and I_3

- (i) Writing the equations in the $a_1x + b_1y + c_1z + d_1 = 0$ form gives:

$$2I_1 + 3I_2 - 4I_3 - 26 = 0$$

$$I_1 - 5I_2 - 3I_3 + 87 = 0$$

$$-7I_1 + 2I_2 + 6I_3 - 12 = 0$$

- (ii) the solution is given by

$$\frac{I_1}{D_{I_1}} = \frac{-I_2}{D_{I_2}} = \frac{I_3}{D_{I_3}} = \frac{-1}{D}$$

where D_{I_1} is the determinant of coefficients obtained by covering up the I_1 column, i.e.

$$\begin{aligned} D_{I_1} &= \begin{vmatrix} 3 & -4 & -26 \\ -5 & -3 & 87 \\ 2 & 6 & -12 \end{vmatrix} \\ &= (3) \begin{vmatrix} -3 & 87 \\ 6 & -12 \end{vmatrix} - (-4) \begin{vmatrix} -5 & 87 \\ 2 & -12 \end{vmatrix} \\ &\quad + (-26) \begin{vmatrix} -5 & -3 \\ 2 & 6 \end{vmatrix} \\ &= 3(-486) + 4(-114) - 26(-24) \\ &= \mathbf{-1290} \end{aligned}$$

$$\begin{aligned} D_{I_2} &= \begin{vmatrix} 2 & -4 & -26 \\ 1 & -3 & 87 \\ -7 & 6 & -12 \end{vmatrix} \\ &= (2)(36 - 522) - (-4)(-12 + 609) \\ &\quad + (-26)(6 - 21) \\ &= -972 + 2388 + 390 \\ &= \mathbf{1806} \end{aligned}$$

$$\begin{aligned} D_{I_3} &= \begin{vmatrix} 2 & 3 & -26 \\ 1 & -5 & 87 \\ -7 & 2 & -12 \end{vmatrix} \\ &= (2)(60 - 174) - (3)(-12 + 609) \\ &\quad + (-26)(2 - 35) \\ &= -228 - 1791 + 858 = \mathbf{-1161} \end{aligned}$$

$$\begin{aligned} \text{and } D &= \begin{vmatrix} 2 & 3 & -4 \\ 1 & -5 & -3 \\ -7 & 2 & 6 \end{vmatrix} \\ &= (2)(-30 + 6) - (3)(6 - 21) \\ &\quad + (-4)(2 - 35) \\ &= -48 + 45 + 132 = \mathbf{129} \end{aligned}$$

Thus

$$\frac{I_1}{-1290} = \frac{-I_2}{1806} = \frac{I_3}{-1161} = \frac{-1}{129}$$

giving

$$I_1 = \frac{-1290}{-129} = \mathbf{10 \text{ mA}},$$

$$I_2 = \frac{1806}{129} = \mathbf{14 \text{ mA}}$$

$$\text{and } I_3 = \frac{1161}{129} = \mathbf{9 \text{ mA}}$$

Now try the following Practice Exercise

Practice Exercise 111 Solving simultaneous equations using determinants (Answers on page 847)

In Problems 1 to 5 use **determinants** to solve the simultaneous equations given.

- $3x - 5y = -17.6$
 $7y - 2x - 22 = 0$
- $2.3m - 4.4n = 6.84$
 $8.5n - 6.7m = 1.23$
- $3x + 4y + z = 10$
 $2x - 3y + 5z + 9 = 0$
 $x + 2y - z = 6$
- $1.2p - 2.3q - 3.1r + 10.1 = 0$
 $4.7p + 3.8q - 5.3r - 21.5 = 0$
 $3.7p - 8.3q + 7.4r + 28.1 = 0$
- $\frac{x}{2} - \frac{y}{3} + \frac{2z}{5} = -\frac{1}{20}$
 $\frac{x}{4} + \frac{2y}{3} - \frac{z}{2} = \frac{19}{40}$
 $x + y - z = \frac{59}{60}$
- In a system of forces, the relationship between two forces F_1 and F_2 is given by:
 $5F_1 + 3F_2 + 6 = 0$
 $3F_1 + 5F_2 + 18 = 0$
Use determinants to solve for F_1 and F_2
- Applying mesh-current analysis to an a.c. circuit results in the following equations:
 $(5 - j4)I_1 - (-j4)I_2 = 100\angle 0^\circ$
 $(4 + j3 - j4)I_2 - (-j4)I_1 = 0$
Solve the equations for I_1 and I_2
- Kirchhoff's laws are used to determine the current equations in an electrical network and show that
 $i_1 + 8i_2 + 3i_3 = -31$
 $3i_1 - 2i_2 + i_3 = -5$
 $2i_1 - 3i_2 + 2i_3 = 6$
Use determinants to find the values of i_1 , i_2 and i_3

- The forces in three members of a framework are F_1 , F_2 and F_3 . They are related by the simultaneous equations shown below.

$$1.4F_1 + 2.8F_2 + 2.8F_3 = 5.6$$

$$4.2F_1 - 1.4F_2 + 5.6F_3 = 35.0$$

$$4.2F_1 + 2.8F_2 - 1.4F_3 = -5.6$$

Find the values of F_1 , F_2 and F_3 using determinants.

- Mesh-current analysis produces the following three equations:

$$20\angle 0^\circ = (5 + 3 - j4)I_1 - (3 - j4)I_2$$

$$10\angle 90^\circ = (3 - j4 + 2)I_2 - (3 - j4)I_1 - 2I_3$$

$$-15\angle 0^\circ - 10\angle 90^\circ = (12 + 2)I_3 - 2I_2$$

Solve the equations for the loop currents I_1 , I_2 and I_3

25.3 Solution of simultaneous equations using Cramer's rule

Cramer's* rule states that if

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

$$\text{then } x = \frac{D_x}{D}, y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$



* **Who was Cramer?** **Gabriel Cramer** (31 July 1704–4 January 1752) was a Swiss mathematician. His articles cover a wide range of subjects including the study of geometric problems, the history of mathematics, philosophy, and the date of Easter. Cramer's most famous book is a work which Cramer modelled on Newton's memoir on cubic curves. To find out more go to www.routledge.com/cw/bird

where $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$D_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

i.e. the x -column has been replaced by the RHS b column,

$$D_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

i.e. the y -column has been replaced by the RHS b column,

$$D_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

i.e. the z -column has been replaced by the RHS b column.

Problem 7. Solve the following simultaneous equations using Cramer's rule.

$$x + y + z = 4$$

$$2x - 3y + 4z = 33$$

$$3x - 2y - 2z = 2$$

(This is the same as Problem 2 and a comparison of methods may be made). Following the above method:

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & -2 \end{vmatrix}$$

$$= 1(6 - (-8)) - 1((-4) - 12)$$

$$+ 1((-4) - (-9)) = 14 + 16 + 5 = 35$$

$$D_x = \begin{vmatrix} 4 & 1 & 1 \\ 33 & -3 & 4 \\ 2 & -2 & -2 \end{vmatrix}$$

$$= 4(6 - (-8)) - 1((-66) - 8)$$

$$+ 1((-66) - (-6)) = 56 + 74 - 60 = 70$$

$$D_y = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 33 & 4 \\ 3 & 2 & -2 \end{vmatrix}$$

$$= 1((-66) - 8) - 4((-4) - 12) + 1(4 - 99)$$

$$= -74 + 64 - 95 = -105$$

$$D_z = \begin{vmatrix} 1 & 1 & 4 \\ 2 & -3 & 33 \\ 3 & -2 & 2 \end{vmatrix}$$

$$= 1((-6) - (-66)) - 1(4 - 99)$$

$$+ 4((-4) - (-9)) = 60 + 95 + 20 = 175$$

Hence

$$x = \frac{D_x}{D} = \frac{70}{35} = 2, \quad y = \frac{D_y}{D} = \frac{-105}{35} = -3$$

and $z = \frac{D_z}{D} = \frac{175}{35} = 5$

Now try the following Practice Exercise

Practice Exercise 112 Solving simultaneous equations using Cramer's rule (Answers on page 847)

1. Repeat problems 3, 4, 5, 7 and 8 of Exercise 110 on page 287, using Cramer's rule.
2. Repeat problems 3, 4, 8 and 9 of Exercise 111 on page 291, using Cramer's rule.

25.4 Solution of simultaneous equations using the Gaussian elimination method

Consider the following simultaneous equations:

$$x + y + z = 4 \quad (1)$$

$$2x - 3y + 4z = 33 \quad (2)$$

$$3x - 2y - 2z = 2 \quad (3)$$

Leaving equation (1) as it is gives:

$$x + y + z = 4 \quad (1)$$

Equation (2) $- 2 \times$ equation (1) gives:

$$0 - 5y + 2z = 25 \quad (2')$$

and equation (3) $- 3 \times$ equation (1) gives:

$$0 - 5y - 5z = -10 \quad (3')$$

Leaving equations (1) and (2') as they are gives:

$$x + y + z = 4 \quad (1)$$

$$0 - 5y + 2z = 25 \quad (2')$$

Equation (3') $-$ equation (2') gives:

$$0 + 0 - 7z = -35 \quad (3'')$$

By appropriately manipulating the three original equations we have deliberately obtained zeros in the positions shown in equations (2') and (3'').

Working backwards, from equation (3''),

$$z = \frac{-35}{-7} = 5$$

from equation (2'),

$$-5y + 2(5) = 25,$$

from which,

$$y = \frac{25 - 10}{-5} = -3$$

and from equation (1),

$$x + (-3) + 5 = 4$$

from which,

$$x = 4 + 3 - 5 = 2$$

(This is the same example as Problems 2 and 7, and a comparison of methods can be made). The

above method is known as the **Gaussian*** elimination method.

We conclude from the above example that if

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The three-step **procedure** to solve simultaneous equations in three unknowns using the **Gaussian elimination method** is:

- (i) Equation (2) $- \frac{a_{21}}{a_{11}} \times$ equation (1) to form equation (2') and equation (3) $- \frac{a_{31}}{a_{11}} \times$ equation (1) to form equation (3').
- (ii) Equation (3') $- \frac{a_{32}}{a_{22}} \times$ equation (2') to form equation (3'').
- (iii) Determine z from equation (3''), then y from equation (2') and finally, x from equation (1).



* **Who was Gauss?** Johann Carl Friedrich Gauss (30 April 1777–23 February 1855) was a German mathematician and physical scientist who contributed significantly to many fields, including number theory, statistics, electrostatics, astronomy and optics. To find out more go to www.routledge.com/cw/bird

Problem 8. A d.c. circuit comprises three closed loops. Applying Kirchhoff's laws to the closed loops gives the following equations for current flow in milliamperes:

$$2I_1 + 3I_2 - 4I_3 = 26 \quad (1)$$

$$I_1 - 5I_2 - 3I_3 = -87 \quad (2)$$

$$-7I_1 + 2I_2 + 6I_3 = 12 \quad (3)$$

Use the Gaussian elimination method to solve for I_1 , I_2 and I_3

(This is the same example as Problem 6 on page 290, and a comparison of methods may be made)

Following the above procedure:

$$1. \quad 2I_1 + 3I_2 - 4I_3 = 26 \quad (1)$$

Equation (2) $-\frac{1}{2} \times$ equation (1) gives:

$$0 - 6.5I_2 - I_3 = -100 \quad (2')$$

Equation (3) $-\frac{-7}{2} \times$ equation (1) gives:

$$0 + 12.5I_2 - 8I_3 = 103 \quad (3')$$

$$2. \quad 2I_1 + 3I_2 - 4I_3 = 26 \quad (1)$$

$$0 - 6.5I_2 - I_3 = -100 \quad (2')$$

Equation (3') $-\frac{12.5}{-6.5} \times$ equation (2') gives:

$$0 + 0 - 9.923I_3 = -89.308 \quad (3'')$$

$$3. \quad \text{From equation (3''),}$$

$$I_3 = \frac{-89.308}{-9.923} = 9 \text{ mA}$$

from equation (2'), $-6.5I_2 - 9 = -100$,

$$\text{from which, } I_2 = \frac{-100 + 9}{-6.5} = 14 \text{ mA}$$

and from equation (1), $2I_1 + 3(14) - 4(9) = 26$,

$$\text{from which, } I_1 = \frac{26 - 42 + 36}{2} = \frac{20}{2} = 10 \text{ mA}$$

Now try the following Practice Exercise

Practice Exercise 113 Solving simultaneous equations using Gaussian elimination (Answers on page 847)

1. In a mass–spring–damper system, the acceleration $\ddot{x}$ m/s², velocity $\dot{x}$ m/s and displacement x m are related by the following simultaneous equations:

$$6.2\ddot{x} + 7.9\dot{x} + 12.6x = 18.0$$

$$7.5\ddot{x} + 4.8\dot{x} + 4.8x = 6.39$$

$$13.0\ddot{x} + 3.5\dot{x} - 13.0x = -17.4$$

By using Gaussian elimination, determine the acceleration, velocity and displacement for the system, correct to 2 decimal places.

2. The tensions, T_1 , T_2 and T_3 in a simple framework are given by the equations:

$$5T_1 + 5T_2 + 5T_3 = 7.0$$

$$T_1 + 2T_2 + 4T_3 = 2.4$$

$$4T_1 + 2T_2 = 4.0$$

Determine T_1 , T_2 and T_3 using Gaussian elimination.

3. Repeat problems 3, 4, 5, 7 and 8 of Exercise 110 on page 287, using the Gaussian elimination method.
4. Repeat problems 3, 4, 8 and 9 of Exercise 111 on page 291, using the Gaussian elimination method.

25.5 Eigenvalues and eigenvectors

In practical applications, such as coupled oscillations and vibrations, equations of the form:

$$A\mathbf{x} = \lambda \mathbf{x}$$

occur, where A is a square matrix and λ (Greek lambda) is a number. Whenever $\mathbf{x} \neq \mathbf{0}$, the values of λ are called the **eigenvalues** of the matrix A ; the corresponding solutions of the equation $A\mathbf{x} = \lambda \mathbf{x}$ are called the **eigenvectors** of A .

Sometimes, instead of the term *eigenvalues*, **characteristic values** or **latent roots** are used. Also, instead of the term *eigenvectors*, **characteristic vectors** is used.

From above, if $A\mathbf{x} = \lambda \mathbf{x}$ then $A\mathbf{x} - \lambda \mathbf{x} = \mathbf{0}$ i.e. $(A - \lambda I)\mathbf{x} = \mathbf{0}$ where I is the unit matrix.

If $\mathbf{x} \neq \mathbf{0}$ then $|A - \lambda I| = 0$

$|A - \lambda I|$ is called the **characteristic determinant** of A and $|A - \lambda I| = 0$ is called the **characteristic equation**. Solving the characteristic equation will give the value(s) of the eigenvalues, as demonstrated in the following worked problems.

Problem 9. Determine the eigenvalues of the matrix $A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$

The eigenvalue is determined by solving the characteristic equation $|A - \lambda I| = 0$

$$\text{i.e. } \left| \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\text{i.e. } \left| \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\text{i.e. } \begin{vmatrix} 3-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

(Given a square matrix, we can get used to going straight to this characteristic equation)

$$\text{Hence, } (3 - \lambda)(1 - \lambda) - (2)(4) = 0$$

$$\text{i.e. } 3 - 3\lambda - \lambda + \lambda^2 - 8 = 0$$

$$\text{and } \lambda^2 - 4\lambda - 5 = 0$$

$$\text{i.e. } (\lambda - 5)(\lambda + 1) = 0$$

from which, $\lambda - 5 = 0$ i.e. $\lambda = 5$ or $\lambda + 1 = 0$ i.e. $\lambda = -1$

(Instead of factorising, the quadratic formula could be used; even electronic calculators can solve quadratic equations.)

Hence, the eigenvalues of the matrix $\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$ are 5 and -1

Problem 10. Determine the eigenvectors of the matrix $A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$

From Problem 9, the eigenvalues of $\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$ are $\lambda_1 = 5$ and $\lambda_2 = -1$

Using the equation $(A - \lambda I)x = 0$ for $\lambda_1 = 5$

$$\text{then } \begin{pmatrix} 3-5 & 4 \\ 2 & 1-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which,

$$-2x_1 + 4x_2 = 0$$

and

$$2x_1 - 4x_2 = 0$$

From either of these two equations, $x_1 = 2x_2$

Hence, whatever value x_2 is, the value of x_1 will be two times greater. Hence the simplest eigenvector is:

$$x_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Using the equation $(A - \lambda I)x = 0$ for $\lambda_2 = -1$

$$\text{then } \begin{pmatrix} 3-(-1) & 4 \\ 2 & 1-(-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which,

$$4x_1 + 4x_2 = 0$$

and

$$2x_1 + 2x_2 = 0$$

From either of these two equations, $x_1 = -x_2$ or $x_2 = -x_1$

Hence, whatever value x_1 is, the value of x_2 will be -1 times greater. Hence the simplest eigenvector is:

$$x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Summarising, $x_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 5$ and $x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = -1$

Problem 11. Determine the eigenvalues of the matrix $A = \begin{pmatrix} 5 & -2 \\ -9 & 2 \end{pmatrix}$

The eigenvalue is determined by solving the characteristic equation $|A - \lambda I| = 0$

$$\text{i.e. } \begin{vmatrix} 5-\lambda & -2 \\ -9 & 2-\lambda \end{vmatrix} = 0$$

$$\text{Hence, } (5 - \lambda)(2 - \lambda) - (-9)(-2) = 0$$

$$\text{i.e. } 10 - 5\lambda - 2\lambda + \lambda^2 - 18 = 0$$

$$\text{and } \lambda^2 - 7\lambda - 8 = 0$$

$$\text{i.e. } (\lambda - 8)(\lambda + 1) = 0$$

from which, $\lambda - 8 = 0$ i.e. $\lambda = 8$ or $\lambda + 1 = 0$ i.e. $\lambda = -1$ (or use the quadratic formula).

Hence, the eigenvalues of the matrix $\begin{pmatrix} 5 & -2 \\ -9 & 2 \end{pmatrix}$ are 8 and -1

Problem 12. Determine the eigenvectors of the matrix $A = \begin{pmatrix} 5 & -2 \\ -9 & 2 \end{pmatrix}$

From Problem 11, the eigenvalues of $\begin{pmatrix} 5 & -2 \\ -9 & 2 \end{pmatrix}$ are $\lambda_1 = 8$ and $\lambda_2 = -1$

Using the equation $(A - \lambda I)x = 0$ for $\lambda_1 = 8$

$$\text{then } \begin{pmatrix} 5-8 & -2 \\ -9 & 2-8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} -3 & -2 \\ -9 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which,

$$-3x_1 - 2x_2 = 0$$

and

$$-9x_1 - 6x_2 = 0$$

From either of these two equations, $3x_1 = -2x_2$ or $x_1 = -\frac{2}{3}x_2$

Hence, if $x_2 = 3$, $x_1 = -2$. Hence the simplest eigenvector is: $x_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

Using the equation $(A - \lambda I)x = 0$ for $\lambda_2 = -1$

$$\text{then } \begin{pmatrix} 5-(-1) & -2 \\ -9 & 2-(-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 6 & -2 \\ -9 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which,

$$6x_1 - 2x_2 = 0$$

and

$$-9x_1 + 3x_2 = 0$$

From either of these two equations, $x_2 = 3x_1$

Hence, if $x_1 = 1$, $x_2 = 3$. Hence the simplest eigenvector is: $x_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Summarising, $x_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 8$ and $x_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = -1$

Problem 13. Determine the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$

The eigenvalue is determined by solving the characteristic equation $|A - \lambda I| = 0$

$$\text{i.e. } \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} = 0$$

Hence, using the top row:

$$\begin{aligned} (1-\lambda)[(-1-\lambda)(-1-\lambda) - (-2)(0)] \\ - 2[6(-1-\lambda) - (-1)(0)] \\ + 1[(6)(-2) - (-1)(-1-\lambda)] = 0 \end{aligned}$$

$$\text{i.e. } (1-\lambda)[1 + \lambda + \lambda + \lambda^2] - 2[-6 - 6\lambda] + 1[-12 - 1 - \lambda] = 0$$

$$\text{i.e. } (1-\lambda)[\lambda^2 + 2\lambda + 1] + 12 + 12\lambda - 13 - \lambda = 0$$

$$\text{and } \lambda^2 + 2\lambda + 1 - \lambda^3 - 2\lambda^2 - \lambda + 12 + 12\lambda - 13 - \lambda = 0$$

$$\text{i.e. } -\lambda^3 - \lambda^2 + 12\lambda = 0$$

$$\text{or } \lambda^3 + \lambda^2 - 12\lambda = 0$$

$$\text{i.e. } \lambda(\lambda^2 + \lambda - 12) = 0$$

$$\text{i.e. } \lambda(\lambda - 3)(\lambda + 4) = 0 \text{ by factorising}$$

from which, $\lambda = 0$, $\lambda = 3$ or $\lambda = -4$

Hence, the eigenvalues of the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$ are 0, 3 and -4

Problem 14. Determine the eigenvectors of the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$

From Problem 13, the eigenvalues of $\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$ are $\lambda_1 = 0$, $\lambda_2 = 3$ and $\lambda_3 = -4$

Using the equation $(A - \lambda I)x = 0$ for $\lambda_1 = 0$

$$\text{then } \begin{pmatrix} 1-0 & 2 & 1 \\ 6 & -1-0 & 0 \\ -1 & -2 & -1-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$x_1 + 2x_2 + x_3 = 0$$

$$6x_1 - x_2 = 0$$

$$-x_1 - 2x_2 - x_3 = 0$$

From the second equation,

$$6x_1 = x_2$$

Substituting in the first equation,

$$x_1 + 12x_1 + x_3 = 0 \quad \text{i.e. } -13x_1 = x_3$$

Hence, when $x_1 = 1$, $x_2 = 6$ and $x_3 = -13$ Hence the simplest eigenvector corresponding to $\lambda_1 = 0$

$$\text{is: } \mathbf{x}_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}$$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_2 = 3$

$$\text{then } \begin{pmatrix} 1-3 & 2 & 1 \\ 6 & -1-3 & 0 \\ -1 & -2 & -1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$-2x_1 + 2x_2 + x_3 = 0$$

$$6x_1 - 4x_2 = 0$$

$$-x_1 - 2x_2 - 4x_3 = 0$$

From the second equation,

$$3x_1 = 2x_2$$

Substituting in the first equation,

$$-2x_1 + 3x_1 + x_3 = 0 \quad \text{i.e. } x_3 = -x_1$$

Hence, if $x_2 = 3$, then $x_1 = 2$ and $x_3 = -2$ Hence the simplest eigenvector corresponding to $\lambda_2 = 3$

$$\text{is: } \mathbf{x}_2 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$

Using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda_2 = -4$

$$\text{then } \begin{pmatrix} 1-(-4) & 2 & 1 \\ 6 & -1-(-4) & 0 \\ -1 & -2 & -1-(-4) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$5x_1 + 2x_2 + x_3 = 0$$

$$6x_1 + 3x_2 = 0$$

$$-x_1 - 2x_2 + 3x_3 = 0$$

From the second equation,

$$x_2 = -2x_1$$

Substituting in the first equation,

$$5x_1 - 4x_1 + x_3 = 0 \quad \text{i.e. } x_3 = -x_1$$

Hence, if $x_1 = -1$, then $x_2 = 2$ and $x_3 = 1$ Hence the simplest eigenvector corresponding to $\lambda_2 =$

$$-4 \text{ is: } \mathbf{x}_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Problem 15. Determine the eigenvalues of the

$$\text{matrix } \mathbf{A} = \begin{pmatrix} 1 & -4 & -2 \\ 0 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

The eigenvalue is determined by solving the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$

$$\text{i.e. } \begin{vmatrix} 1-\lambda & -4 & -2 \\ 0 & 3-\lambda & 1 \\ 1 & 2 & 4-\lambda \end{vmatrix} = 0$$

Hence, using the top row:

$$(1-\lambda)[(3-\lambda)(4-\lambda) - (2)(1)] - (-4)[0 - (1)(1)] - 2[0 - 1(3-\lambda)] = 0$$

$$\text{i.e. } (1-\lambda)[12 - 3\lambda - 4\lambda + \lambda^2 - 2] + 4[-1] - 2[-3 + \lambda] = 0$$

$$\text{i.e. } (1-\lambda)[\lambda^2 - 7\lambda + 10] - 4 + 6 - 2\lambda = 0$$

$$\text{and } \lambda^2 - 7\lambda + 10 - \lambda^3 + 7\lambda^2 - 10\lambda - 4 + 6 - 2\lambda = 0$$

i.e. $-\lambda^3 + 8\lambda^2 - 19\lambda + 12 = 0$
 or $\lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0$

To solve a cubic equation, the **factor theorem** (see Chapter 1) may be used. (Alternatively, electronic calculators can solve such cubic equations.)

Let $f(\lambda) = \lambda^3 - 8\lambda^2 + 19\lambda - 12$

If $\lambda = 1$, then $f(1) = 1^3 - 8(1)^2 + 19(1) - 12 = 0$

Since $f(1) = 0$ then $(\lambda - 1)$ is a factor.

If $\lambda = 2$, then $f(2) = 2^3 - 8(2)^2 + 19(2) - 12 \neq 0$

If $\lambda = 3$, then $f(3) = 3^3 - 8(3)^2 + 19(3) - 12 = 0$

Since $f(3) = 0$ then $(\lambda - 3)$ is a factor.

If $\lambda = 4$, then $f(4) = 4^3 - 8(4)^2 + 19(4) - 12 = 0$

Since $f(4) = 0$ then $(\lambda - 4)$ is a factor.

Thus, since $\lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0$

then $(\lambda - 1)(\lambda - 3)(\lambda - 4) = 0$

from which, $\lambda = 1$ or $\lambda = 3$ or $\lambda = 4$

Hence, the eigenvalues of the matrix

$\begin{pmatrix} 1 & -4 & -2 \\ 0 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ **are 1, 3 and 4**

Problem 16. Determine the eigenvectors of the

matrix $A = \begin{pmatrix} 1 & -4 & -2 \\ 0 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$

From Problem 15, the eigenvalues of $\begin{pmatrix} 1 & -4 & -2 \\ 0 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$

are $\lambda_1 = 1$, $\lambda_2 = 3$ and $\lambda_3 = 4$

Using the equation $(A - \lambda I)x = 0$ for $\lambda_1 = 1$

then $\begin{pmatrix} 1-1 & -4 & -2 \\ 0 & 3-1 & 1 \\ 1 & 2 & 4-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

i.e.

$$\begin{pmatrix} 0 & -4 & -2 \\ 0 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$-4x_2 - 2x_3 = 0$$

$$2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + 3x_3 = 0$$

From the first two equations,

$$x_3 = -2x_2 \quad (\text{i.e. if } x_2 = 1, x_3 = -2)$$

From the last equation,

$$x_1 = -2x_2 - 3x_3 \quad \text{i.e. } x_1 = -2x_2 - 3(-2x_2)$$

i.e.

$$x_1 = 4x_2 \quad (\text{i.e. if } x_2 = 1, x_1 = 4)$$

Hence the simplest eigenvector corresponding to $\lambda_1 = 1$

is: $\mathbf{x}_1 = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$

Using the equation $(A - \lambda I)x = 0$ for $\lambda_2 = 3$

then $\begin{pmatrix} 1-3 & -4 & -2 \\ 0 & 3-3 & 1 \\ 1 & 2 & 4-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

i.e.

$$\begin{pmatrix} -2 & -4 & -2 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which

$$-2x_1 - 4x_2 - 2x_3 = 0$$

$$x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

Since $x_3 = 0$, $x_1 = -2x_2$ (i.e. if $x_2 = 1$, $x_1 = -2$)

Hence the simplest eigenvector corresponding to $\lambda_2 = 3$

is: $\mathbf{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

Using the equation $(A - \lambda I)x = 0$ for $\lambda_3 = 4$

then $\begin{pmatrix} 1-4 & -4 & -2 \\ 0 & 3-4 & 1 \\ 1 & 2 & 4-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

i.e.

$$\begin{pmatrix} -3 & -4 & -2 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which,

$$-3x_1 - 4x_2 - 2x_3 = 0$$

$$-x_2 + x_3 = 0$$

$$x_1 + 2x_2 = 0$$

from which, $x_3 = x_2$ and $x_1 = -2x_2$ (i.e. if $x_2 = 1$, $x_1 = -2$ and $x_3 = 1$)

Hence the simplest eigenvector corresponding to $\lambda_3 = 4$

is: $\mathbf{x}_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

Now try the following Practice Exercise

Practice Exercise 114 Eigenvalues and eigenvectors (Answers on page 847)

For each of the following matrices, determine their
(a) eigenvalues (b) eigenvectors

1. $\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$ 2. $\begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix}$ 3. $\begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$

4. $\begin{pmatrix} -1 & -1 & 1 \\ -4 & 2 & 4 \\ -1 & 1 & 5 \end{pmatrix}$ 5. $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

6. $\begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ 7. $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$

For fully worked solutions to each of the problems in Practice Exercises 110 to 114 in this chapter,
go to the website:

www.routledge.com/cw/bird

